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Differential Geometry of Spacetime Tangent Bundle

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Conditions are investigated under which the Levi-Civita connection of the spacetime tangent bundle corresponds to that of a generic tangent bundle of a Finsler manifold. Also, requirements are specified for the spacetime tangent bundle to be almost complex or Kählerian.

1. SPACETIME TANGENT BUNDLE

Both string theory and the quantum mechanics of the vacuum polarization in accelerated frames determine a universal upper limit on allowable proper acceleration relative to the vacuum (Brandt, 1983, 1984*a*, 1989*a*, 1991*a*,*b*; Sakai, 1986; Parentani and Potting, 1989). If the limiting acceleration is universal, then it must apply invariantly for all observers. The latter requirement defines the maximal-acceleration invariant phase space as a fiber bundle in which spacetime is the base manifold and four-velocity space is the fiber manifold (Brandt, 1984*b*, 1987*a*,*b*, 1989*a*–*c*, 1991*a*,*b*). In a coordinate basis, the implied structure of the bundle metric G_{AB} is that of the diagonal lift (Yano and Ishihara, 1973) of the spacetime metric $g_{\mu\nu}$, namely,

$$G_{AB}=\begin{pmatrix} g_{\mu\nu}+g_{\alpha\beta}A^{\alpha}{}_{\mu}A^{\beta}{}_{\nu} & A_{n\mu} \\ A_{m\nu} & g_{mn} \end{pmatrix} \tag{1}$$

where $A^{\mu}{}_{\nu}$ is the gauge potential,

$$A^{\mu}{}_{\nu}=\rho_0 v^{\lambda}\Gamma^{\mu}{}_{\lambda\nu} \tag{2}$$

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and $\Gamma^{\mu}{}_{\lambda\nu}$ is the spacetime affine connection. A point in the bundle manifold has coordinates

$$\begin{aligned}\{x^M; M=0,1,\ldots,7\} &= \{x^\mu, x^m; \mu=0,1,2,3; m=4,5,6,7\}\\ &\equiv \{x^\mu, \rho_0 v^\mu; \mu=0,1,2,3\}\end{aligned} \tag{3}$$

where x^μ and v^μ are the spacetime and four-velocity coordinates, respectively. Greek indices referring to spacetime range from 0 to 3, lower case Latin indices referring to four-velocity space range from 4 to 7, and upper case Latin indices referring to a point in the bundle range from 0 to 7. Any lower case Latin index n appearing in a canonical spacetime tensor or connection is defined to be $n-4$ implicitly. The length ρ_0 is of the order of the Planck length and is given by

$$\rho_0 = c^2/a_0 = (\hbar G/c^3)^{1/2}/2\pi\alpha \tag{4}$$

where a_0 is the maximal proper acceleration relative to the vacuum, c is the velocity of light in vacuum, $\hbar$ is Planck's constant divided by 2π, G is the universal gravitational constant, and α is a dimensionless number of order unity (Brandt, 1983, 1984*a*, 1989*a*, 1991*a*,*b*).

In an anholonomic basis adapted to the affine connection (Brandt, 1989*a*, 1987*b*; Yano and Ishihara, 1973) the bundle metric has the simple block diagonal form

$$G_{AB} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & g_{mn} \end{pmatrix} \tag{5}$$

and the Levi-Civita connection coefficients ${}^{(8)}\Gamma^M{}_{AB}$ of the bundle manifold are given by (Brandt, 1989*a*)

$${}^{(8)}\Gamma^\mu{}_{\alpha\beta} = \left\{{}^{\mu}{}_{\alpha\beta}\right\} = \tfrac{1}{2} g^{\mu\lambda}(g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda}) \tag{6}$$

$${}^{(8)}\Gamma^\mu{}_{ab} = {}^{(8)}\Gamma^\mu{}_{ba} = \tfrac{1}{2}(F_{ba}{}^\mu + \Pi^\mu{}_{ba} + \Pi_{ab}{}^\mu) \tag{7}$$

$${}^{(8)}\Gamma^\mu{}_{ab} = -\tfrac{1}{2}(T_a{}^\mu{}_b + T_b{}^\mu{}_a) \tag{8}$$

$${}^{(8)}\Gamma^m{}_{\alpha\beta} = \tfrac{1}{2}(F^m{}_{\alpha\beta} - \Pi_\alpha{}^m{}_\beta - \Pi_\beta{}^m{}_\alpha) \tag{9}$$

$${}^{(8)}\Gamma^m{}_{ab} = \tfrac{1}{2}(T_{ba}{}^m + T^m{}_{ab}) \tag{10}$$

$${}^{(8)}\Gamma^m{}_{ba} = \left\{{}^{m}{}_{ba}\right\} + \tfrac{1}{2}(T_{ba}{}^m - T^m{}_{ab}) \tag{11}$$

$${}^{(8)}\Gamma^m{}_{ab} = \Pi^m{}_{ab} = \tfrac{1}{2}\rho_0^{-1} g^{mn}(\partial/\partial v^a g_{nb} + \partial/\partial v^b g_{na} - \partial/\partial v^n g_{ab}) \tag{12}$$

Here $\{{}^{\mu}{}_{\alpha\beta}\}$ and $\Pi^{\mu}{}_{\alpha\beta}$ are the Christoffel symbols in spacetime and four-velocity space, respectively, and in the anholonomic basis, the following notation is implicit:

$$_{,\mu} \equiv \partial/\partial x^{\mu} - \rho_0^{-1} A^{\beta}{}_{\mu}\, \partial/\partial v^{\beta} \tag{13}$$

In addition, $F^{\alpha}{}_{\mu\nu}$ is the gauge curvature field given by

$$F^{\alpha}{}_{\mu\nu} = \rho_0 v^{\lambda} R^{\alpha}{}_{\lambda\mu\nu} \tag{14}$$

where $R^{\alpha}{}_{\lambda\mu\nu}$ is the spacetime Riemann curvature tensor. The field $T^{\beta}{}_{\mu\nu}$ is given by

$$T^{\beta}{}_{\mu\nu} = \{{}^{\beta}{}_{\mu\nu}\} - \rho_0^{-1}\, \partial/\partial v^{\nu}\, A^{\beta}{}_{\mu} \tag{15}$$

In general, the spacetime base manifold of the maximal-acceleration invariant fiber bundle is non-Riemannian. The physical interpretations and applications of such a general spacetime bundle manifold remain to be explored. Recently, as a very special case, a Riemannian spacetime manifold was considered for the base manifold, and it was shown that in this case the bundle manifold is the associated tangent bundle, and the natural lift of a spacetime geodesic is also a geodesic in the spacetime tangent bundle (Yano and Ishihara, 1973; Yano and Davies, 1963; Brandt, 1991*c*). Conversely, if the natural lift of a curve in Riemannian spacetime is a geodesic in the spacetime tangent bundle, then either (a) the spacetime curve is a geodesic, or (b) its proper acceleration is a nonvanishing constant, and the Riemann sectional curvature with respect to the section determined by the osculating plane of the spacetime curve is constant at every point and given by the inverse square of ρ_0. Also, a Riemannian Schwarzschild-like spacetime was considered which is a solution following from an appropriate action defined on the spacetime tangent bundle (Brandt, 1991*a*). Possible modifications were calculated to the canonical red shift formula for a Schwarzschild spacetime. It is of interest to consider more general base manifolds, such as Finsler spacetime (Bejancu, 1990).

2. FINSLER SPACETIME

If the spacetime manifold is a Finsler manifold, then it has the following form:

$$g_{\mu\nu}(x, v) = \tfrac{1}{2}\, \partial^2/\partial v^{\mu}\, \partial v^{\nu}\, L^2(x, v) \tag{16}$$

where $L(x, v)$ is the fundamental function, a scalar on the spacetime tangent bundle (Yano and Ishihara, 1973; Bejancu, 1990). The Finsler spacetime metric is also homogeneous of degree zero, from which it follows that

$$v^{\alpha}\,\partial/\partial v^{\alpha}\,g_{\mu\nu}=0 \tag{17}$$

Also, from equation (16) it follows that

$$\partial/\partial v^{\alpha}\,g_{\mu\nu}=\partial/\partial v^{\mu}\,g_{\alpha\nu} \tag{18}$$

Therefore, substituting equation (18) in equation (17), one obtains

$$v^{\alpha}\,\partial/\partial v^{\alpha}\,g_{\mu\nu}=v^{\alpha}\,\partial/\partial v^{\mu}\,g_{\alpha\nu}=0 \tag{19}$$

or equivalently

$$v^{\alpha}\Pi_{(\mu\alpha\nu)}=v^{\alpha}\Pi_{(\alpha\mu\nu)}=0 \tag{20}$$

Here and throughout, the following notations are employed:

$$T^{\cdot\cdot}{}_{..(\mu\nu).}\equiv T^{\cdot\cdot}{}_{..\mu\nu.}+T^{\cdot\cdot}{}_{..\nu\mu.}$$

and

$$T^{\cdot\cdot}{}_{..[\mu\nu].}\equiv T^{\cdot\cdot}{}_{..\mu\nu.}-T^{\cdot\cdot}{}_{..\nu\mu.}$$

If the spacetime affine connection $\Gamma^{\mu}{}_{\alpha\beta}$ is of the Levi-Civita form, namely,

$$\Gamma^{\mu}{}_{\alpha\beta}=\left\{{}^{\mu}{}_{\alpha\beta}\right\} \tag{21}$$

and the spacetime manifold is Finslerian, then equations (6)–(12) are readily shown to be of the same form as the well-known Levi-Civita connection coefficients for a generic tangent bundle of a Finsler manifold [equations (3.12a)–(3.12h) of Yano and Davies (1963)]. The connection coefficients are consistent with Cartan's theory of Finsler space, provided the gauge curvature field $F^{\mu}{}_{\alpha\beta}$ is vanishing. Furthermore, if the spacetime metric is independent of the four-velocity, then the coefficients reduce to the form corresponding to a tangent bundle of a Riemannian base manifold (Yano and Ishihara, 1973; Yano and Davies, 1963; Brandt, 1991*c*).

3. ALMOST COMPLEX STRUCTURE

To further characterize the differential geometry of the spacetime tangent bundle, it is of interest to consider the one-form ω defined in the spacetime tangent bundle by

$$\omega=\rho_0 v_{\mu}\,dx^{\mu}=\rho_0 g_{\mu\nu}v^{\nu}\,dx^{\mu} \tag{22}$$

Its exterior differential is readily shown to be

$$d\omega=\tfrac{1}{2}J_{AB}\,dx^{A}\wedge dx^{B} \tag{23}$$

where

$$J_{AB}=\begin{pmatrix}\rho_0 v^\lambda\{_{[\beta a]\lambda}\} & -g_{ab}-\rho_0\Pi_{(ab\lambda)}v^\lambda \\ g_{\alpha\beta}+\rho_0\Pi_{(\beta\alpha\lambda)}v^\lambda & 0\end{pmatrix} \tag{24}$$

If the spacetime connection has the Levi-Civita form and the spacetime is Finslerian, then using equations (1), (21), and (20), one obtains

$$J_A{}^B=\begin{pmatrix}A^\beta{}_\alpha & -\delta_a{}^b-A_{\kappa a}A^{b\kappa} \\ \delta_a{}^\beta & -A^b{}_a\end{pmatrix} \tag{25}$$

Next one verifies that

$$J_A{}^D J_D{}^B=-\delta_A{}^B \tag{26}$$

which is the requirement for $J_A{}^B$ to be an almost complex structure (Yano and Ishihara, 1973; Yano and Davies, 1963: Yano, 1965). Thus, the spacetime tangent bundle of a Finsler spacetime manifold is almost complex, with almost complex structure given by equation (25).

In the anholonomic frame adapted to the spacetime affine connection, the almost complex structure, equation (25), for a Finsler spacetime manifold becomes

$$J_A{}^B=\begin{pmatrix}0 & -\delta_a{}^b \\ \delta_a{}^\beta & 0\end{pmatrix} \tag{27}$$

Next, it is of interest to consider $\nabla_E J_A{}^B$, where ∇_E is the covariant derivative involving the Levi-Civita bundle connection ${}^{(8)}\Gamma^A{}_{BC}$. Using equations (6)–(12) and (27), we find that the components of $\nabla_E J_A{}^B$ reduce to

$$\nabla_\varepsilon J_a{}^\beta=\tfrac{1}{2}(F_a{}^\beta{}_\varepsilon-F^\beta{}_{a\varepsilon}+\Pi_a{}^\beta{}_\varepsilon-\Pi^\beta{}_{a\varepsilon}) \tag{28}$$

$$\nabla_\varepsilon J_a{}^b=\tfrac{1}{2}(T^b{}_{\varepsilon a}-T_{a\varepsilon}{}^b) \tag{29}$$

$$\nabla_\varepsilon J_a{}^\beta=\tfrac{1}{2}(T^\beta{}_{\varepsilon a}-T_{a\varepsilon}{}^\beta) \tag{30}$$

$$\nabla_\varepsilon J_a{}^b=-\tfrac{1}{2}(F_a{}^b{}_\varepsilon-F^b{}_{a\varepsilon}+\Pi_a{}^b{}_\varepsilon-\Pi^b{}_{a\varepsilon}) \tag{31}$$

$$\nabla_e J_a{}^\beta=\tfrac{1}{2}(T_e{}^\beta{}_a-T_{ea}{}^\beta+T_a{}^\beta{}_e-T^\beta{}_{ae}) \tag{32}$$

$$\nabla_e J_a{}^b=\tfrac{1}{2}(F_{ea}{}^b+\Pi_a{}^b{}_e-\Pi^b{}_{ea}) \tag{33}$$

$$\nabla_e J_a{}^\beta=\tfrac{1}{2}(F_{ea}{}^\beta+\Pi_a{}^\beta{}_e-\Pi^\beta{}_{ea}) \tag{34}$$

$$\nabla_e J_a{}^b=-\tfrac{1}{2}(T_e{}^b{}_a-T_{ea}{}^b+T_a{}^b{}_e-T^b{}_{ae}) \tag{35}$$

For the Finsler spacetime manifold, all possible contributions in equations (28)–(35) involving combinations of $\Pi^{\mu}{}_{\alpha\beta}$ and $T^{\mu}{}_{\alpha\beta}$ can be shown, using equations (15)–(21), to be vanishing. It follows that if the spacetime manifold is Finslerian and the gauge curvature field $F^{\mu}{}_{\alpha\beta}$ is vanishing, then equations (28)–(35) are also vanishing, and one concludes that (Yano and Ishihara, 1973; Yano and Davies, 1963)

$$\nabla_E J_A{}^B = 0 \tag{36}$$

Equation (36) is the condition that the spacetime tangent bundle be Kählerian (Yano and Ishihara, 1973; Yano and Davies, 1963; Yano, 1965).

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